

# GROUP ANALYSIS OF NONLINEAR INTERNAL WAVES IN OCEANS

## II: The symmetries and rotationally invariant solution<sup>1</sup>

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**Abstract.** The maximal group of Lie point symmetries of a system of nonlinear equations used in geophysical fluid dynamics is presented. The Lie algebra of this group is infinite-dimensional and involves three arbitrary functions of time. The invariant solution under the rotation and dilation is constructed. Qualitative analysis of the invariant solution is provided and the energy of this solution is presented.

*Keywords:* Geophysical fluid dynamics, Symmetries, Infinite Lie algebra, Invariant solution.

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# 1 Introduction

This is a continuation of the paper [1]. We present here the Lie algebra of the maximal group of Lie point symmetries for system nonlinear equations

$$\Delta\psi_t - g\rho_x - fv_z = \psi_x\Delta\psi_z - \psi_z\Delta\psi_x, \quad (1.1)$$

$$v_t + f\psi_z = \psi_x v_z - \psi_z v_x, \quad (1.2)$$

$$\rho_t + \frac{N^2}{g}\psi_x = \psi_x\rho_z - \psi_z\rho_x \quad (1.3)$$

used in geophysical fluid dynamics, e.g. for investigating internal waves in uniformly stratified incompressible fluids (oceans). Here  $g, f, N$  are constants and  $\Delta$  is the two-dimensional Laplacian:

$$\Delta = D_x^2 + D_z^2.$$

## 2 Symmetries

### 2.1 General case

The point symmetries of Eqs. (1.1)-(1.3) have been computed with the help of DIMSYM 2.3 package. The maximal admitted Lie point transformation group is infinite for arbitrary constants  $f$  and  $N$ . If  $f \neq 0$ , the group is generated by the infinite-dimensional Lie algebra spanned by the following operators:

$$\begin{aligned} X_1 &= \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial \rho}, \quad X_3 = a(t)\frac{\partial}{\partial \psi}, \quad X_4 = \frac{\partial}{\partial t}, \\ X_5 &= b(t) \left[ \frac{\partial}{\partial x} - f \frac{\partial}{\partial v} \right] + b'(t)z \frac{\partial}{\partial \psi}, \\ X_6 &= c(t) \left[ \frac{\partial}{\partial z} + \frac{N^2}{g} \frac{\partial}{\partial \rho} \right] - c'(t)x \frac{\partial}{\partial \psi}, \\ X_7 &= x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}, \\ X_8 &= t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} + 3\psi \frac{\partial}{\partial \psi} - 2fx \frac{\partial}{\partial v} + 2\frac{N^2}{g}z \frac{\partial}{\partial \rho}, \\ X_9 &= z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} - \frac{1}{f} [g\rho + (f^2 - N^2)z] \frac{\partial}{\partial v} + \frac{1}{g} [fv + (f^2 - N^2)x] \frac{\partial}{\partial \rho}. \end{aligned} \quad (2.1)$$

Here  $a(t)$ ,  $b(t)$  and  $c(t)$  are arbitrary functions of time  $t$ .

**Remark 2.1.** The presence of the arbitrary functions  $a(t), b(t), c(t)$  in the symmetry Lie algebra is a characteristic property of incompressible fluids ([2], see also [3]). Namely, the operator  $X_3$  generates the group transformation  $\bar{\psi} = \psi + \varepsilon_3 a(t)$  of the stream function  $\psi$ , where  $\varepsilon_3$  is the group parameter. The invariance of fluid flows under this transformation is quite obvious because the velocity vector  $(\psi_z, v, -\psi_x)$  is invariant under this transformation. The operators  $X_5, X_6$  express the invariance under the generalization  $\bar{x} = x + \varepsilon_5 b(t), \bar{z} = z + \varepsilon_6 c(t)$  of the coordinate translations and the Galilean transformations. They provide a *generalized relativity principle* for the Euler equations in terms of conservation laws (see [4], Section 25.3).

## 2.2 The case $f = 0$

In order to include the special case  $f = 0$ , we multiply the operator  $X_9$  by the constant  $f$  and consider the operator

$$X'_9 = f \left[ z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} \right] - [g\rho + (f^2 - N^2)z] \frac{\partial}{\partial v} + \frac{f}{g} [fv + (f^2 - N^2)x] \frac{\partial}{\partial \rho}.$$

Then we let  $f = 0$  and obtain the operator

$$X'_9 = -[g\rho - N^2 z] \frac{\partial}{\partial v}$$

admitted by Eqs. (1.1)-(1.3) with  $f = 0$ . The solution of the determining equations shows that  $X'_9$  is a particular case of a more general symmetry involving an arbitrary function of two variables. Namely, the system (1.1)-(1.3) with  $f = 0$  admits the infinite-dimensional Lie algebra spanned by the following operators:

$$\begin{aligned} X_1 &= h(v, g\rho - N^2 z) \frac{\partial}{\partial v}, \quad X_2 = \frac{\partial}{\partial \rho}, \quad X_3 = a(t) \frac{\partial}{\partial \psi}, \quad X_4 = \frac{\partial}{\partial t}, \\ X_5 &= b(t) \frac{\partial}{\partial x} + b'(t) z \frac{\partial}{\partial \psi}, \\ X_6 &= c(t) \left[ \frac{\partial}{\partial z} + \frac{N^2}{g} \frac{\partial}{\partial \rho} \right] - c'(t) x \frac{\partial}{\partial \psi}, \\ X_7 &= x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + v \frac{\partial}{\partial v} + \rho \frac{\partial}{\partial \rho} + 2\psi \frac{\partial}{\partial \psi}, \\ X_8 &= t \frac{\partial}{\partial t} + 2x \frac{\partial}{\partial x} + 2z \frac{\partial}{\partial z} + 3\psi \frac{\partial}{\partial \psi} + 2 \frac{N^2}{g} z \frac{\partial}{\partial \rho}, \end{aligned} \tag{2.2}$$

where  $h(v, g\rho - N^2 z)$  is an arbitrary function of two variables. The operator  $X_1$  in (2.1) is obtained from the operator  $X_1$  in (2.2) by taking  $h = 1$ .

### 3 Invariant solution based on rotations and dilations

#### 3.1 The invariants

We will investigate here the invariant solutions with respect to the dilations and rotations with the generators  $X_7$  and  $X_9$ . Let us introduce the notation

$$v_* = fv, \quad u = g\rho, \quad \alpha = f^2 - N^2 \quad (3.1)$$

and write the operators  $X_7, X_9$  in the form

$$\begin{aligned} X_7 &= x \frac{\partial}{\partial x} + z \frac{\partial}{\partial z} + u \frac{\partial}{\partial u} + v_* \frac{\partial}{\partial v_*} + 2\psi \frac{\partial}{\partial \psi}, \\ X_9 &= z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} + (v_* + \alpha x) \frac{\partial}{\partial u} - (u + \alpha z) \frac{\partial}{\partial v_*}. \end{aligned} \quad (3.2)$$

The operators (3.2) coincide with the operators (3.17) from [5],

$$\begin{aligned} X_1 &= x \frac{\partial}{\partial x} + y \frac{\partial}{\partial y} + u \frac{\partial}{\partial u} + v \frac{\partial}{\partial v} + kw \frac{\partial}{\partial w}, \\ X_2 &= y \frac{\partial}{\partial x} - x \frac{\partial}{\partial y} + (v + \alpha x + \beta y) \frac{\partial}{\partial u} - (u - \beta x + \alpha y) \frac{\partial}{\partial v}, \end{aligned}$$

with  $k = 2$  and  $\beta = 0$  upon identifying  $v$  with  $v_*$  and  $y$  with  $z$ . Hence, a basis of invariants for the operators (3.2) contains the time  $t$  and the invariants (3.20) from [5] which have now the form

$$\begin{aligned} J_1 &= \frac{1}{x^2 + z^2} (xu + zv_* + \alpha xz), \\ J_2 &= \frac{1}{x^2 + y^2} \left( xv_* - zu + \frac{\alpha}{2}(x^2 - z^2) \right), \\ J_3 &= \frac{\psi}{x^2 + y^2}. \end{aligned}$$

It is more convenient for our purposes to use, instead of these invariants, the equivalent equations (3.19) from [5] which are written now as follows:

$$\begin{aligned} u &= J_1 x - \left( J_2 + \frac{\alpha}{2} \right) z, \\ v_* &= J_1 z + \left( J_2 - \frac{\alpha}{2} \right) x, \\ \psi &= (x^2 + z^2) J_3. \end{aligned} \quad (3.3)$$

### 3.2 Candidates for the invariant solution

Knowledge of a symmetry algebra allows one to obtain particular exact solutions to differential equations in question. These kind of solutions were considered by S. Lie [6]. They are known today as group invariant solutions (briefly *invariant solutions*) and widely used in the modern literature, particularly in investigating nonlinear differential equations.

The general form of regular invariant solutions is obtained from Eqs. (3.3) by setting

$$J_1 = R(t), \quad J_2 = V(t), \quad J_3 = \phi(t)$$

with undetermined functions  $R(t)$ ,  $V(t)$ ,  $\phi(t)$ . Invoking the notation (3.1) we arrive at the following general form of candidates for the invariant solution with respect to the dilations and rotations with the generators  $X_7$  and  $X_9$  from (2.1):

$$\begin{aligned} v &= \frac{1}{f} \left[ R(t) z + V(t) x + \frac{N^2 - f^2}{2} x \right], \\ \rho &= \frac{1}{g} \left[ R(t) x - V(t) z + \frac{N^2 - f^2}{2} z \right], \\ \psi &= (x^2 + z^2) \phi(t). \end{aligned} \tag{3.4}$$

**Remark 3.1.** Solving the Lie equations for the operator  $X_9$  from (3.2) and using the notation (3.1), one can verify that the operator  $X_9$  from (2.1),

$$X_9 = z \frac{\partial}{\partial x} - x \frac{\partial}{\partial z} - \frac{1}{f} [g\rho + (f^2 - N^2)z] \frac{\partial}{\partial v} + \frac{1}{g} [fv + (f^2 - N^2)x] \frac{\partial}{\partial \rho},$$

generates the following one-parameter transformation group with the parameter  $\varepsilon$ :

$$\begin{aligned} \bar{x} &= x \cos \varepsilon + z \sin \varepsilon, \quad \bar{z} = z \cos \varepsilon - x \sin \varepsilon, \\ g\bar{\rho} &= g\rho \cos \varepsilon + fv \sin \varepsilon - (N^2 - f^2) x \sin \varepsilon, \\ f\bar{v} &= fv \cos \varepsilon - g\rho \sin \varepsilon + (N^2 - f^2) z \sin \varepsilon, \\ \bar{t} &= t, \quad \bar{\psi} = \psi. \end{aligned} \tag{3.5}$$

One can verify by inspection that the transformations (3.5) leave invariant

Eqs. (3.4):

$$\begin{aligned}\bar{v} &= \frac{1}{f} \left[ R(t) \bar{z} + V(t) \bar{x} + \frac{N^2 - f^2}{2} \bar{x} \right], \\ \bar{\rho} &= \frac{1}{g} \left[ R(t) \bar{x} - V(t) \bar{z} + \frac{N^2 - f^2}{2} \bar{z} \right], \\ \bar{\psi} &= (\bar{x}^2 + \bar{z}^2) \phi(t).\end{aligned}$$

### 3.3 Construction of the invariant solution

It remains to determine the functions  $R(t)$ ,  $V(t)$ ,  $\phi(t)$  by substituting the expressions (3.4) for  $\rho$ ,  $v$ ,  $\psi$  in Eqs. (1.1)-(1.3).

Differentiating (3.4) we obtain:

$$\begin{aligned}v_t &= \frac{1}{f} [R' z + V' x], \quad v_x = \frac{1}{f} \left[ \frac{N^2 - f^2}{2} + V \right], \quad v_z = \frac{1}{f} R, \\ \rho_t &= \frac{1}{g} [R' x - V' z], \quad \rho_x = \frac{1}{g} R, \quad \rho_z = \frac{1}{g} \left[ \frac{N^2 - f^2}{2} - V \right], \\ \psi_t &= (x^2 + z^2) \phi', \quad \psi_x = 2x \phi, \quad \psi_z = 2z \phi, \quad \Delta \psi_t = 4\phi'.\end{aligned}\tag{3.6}$$

Substitution of (3.6) in Eqs. (1.1)-(1.3) yields:

$$2\phi' - R = 0,\tag{3.7}$$

$$[V' - 2R\phi] x + [R' + 2V\phi + (N^2 + f^2)\phi] z = 0,\tag{3.8}$$

$$[R' + 2V\phi + (N^2 + f^2)\phi] x - [V' - 2R\phi] z = 0.\tag{3.9}$$

Since  $V, R, \phi$  depend only on  $t$ , Eq. (3.8) implies that

$$V' - 2R\phi = 0\tag{3.10}$$

and

$$R' + 2V\phi + (N^2 + f^2)\phi = 0.\tag{3.11}$$

Eq. (3.9) is satisfied due to Eqs. (3.10), (3.11). Hence, Eqs. (1.1)-(1.3) are reduced to Eqs. (3.7), (3.10), (3.11).

Let us write Eq. (3.7) in the form

$$R = 2\phi'.\tag{3.12}$$

Substitution of the expression for  $R$  into Eq. (3.10) yields  $V' = 4\phi\phi'$ , whence upon integration

$$V = 2\phi^2 + A, \quad A = \text{const.} \quad (3.13)$$

Finally, substituting Eqs. (3.12) and (3.13) in Eq. (3.11) we obtain the following nonlinear second-order ordinary differential equation for  $\phi(t)$ :

$$\phi'' + 2\phi^3 + \left(A + \frac{f^2 + N^2}{2}\right)\phi = 0. \quad (3.14)$$

Thus, we have arrived at the following result.

**Theorem 3.1.** The solutions of the system (1.1)-(1.3) that are invariant with respect to the dilations and rotations with the generators  $X_7$  and  $X_9$  from (2.1) are given by

$$\begin{aligned} v &= \frac{1}{f} \left[ \left( 2\phi^2(t) + A + \frac{N^2 - f^2}{2} \right) x + 2\phi'(t)z \right], \\ \rho &= \frac{1}{g} \left[ 2\phi'(t)x - \left( 2\phi^2(t) + A - \frac{N^2 - f^2}{2} \right) z \right], \\ \psi &= (x^2 + z^2) \phi(t), \end{aligned} \quad (3.15)$$

where  $\phi(t)$  is defined by the differential equation (3.14) and  $A$  is an arbitrary constant.

### 3.4 Qualitative analysis of the invariant solution

One can integrate Eq. (3.14) once, e.g., upon multiplying by  $2\phi'$  and obtain

$$\phi'^2 + \phi^4 + \left(A + \frac{f^2 + N^2}{2}\right)\phi^2 = \text{const.} \quad (3.16)$$

We will analyze the behavior of the solutions to Eq. (3.16) under the assumption that the expression in the parentheses is a non-negative constant which we denote by  $K$ :

$$K = A + \frac{f^2 + N^2}{2}, \quad K \geq 0, \quad (3.17)$$

and write Eq. (3.16) in the form

$$\phi'^2 + \phi^4 + K\phi^2 = B^2, \quad B = \text{const.}, \quad (3.18)$$

or solving for  $\phi'$  :

$$\phi' = \pm \sqrt{B^2 - \phi^4 - K\phi^2}. \quad (3.19)$$

Note that  $\phi(t) = 0$  solves Eq. (3.14). Let us turn to Eq. (3.19). When  $\phi$  is small, i.e. close to the trivial solution  $\phi(t) = 0$ , then

$$B^2 - \phi^4 - K\phi^2 \approx B^2$$

and hence  $\phi'$  is close to the constant value

$$\phi' \approx \pm B.$$

When  $\phi(t)$  varies according to Eq. (3.14), then  $|\phi'|$  decreases since

$$B^2 - \phi^4 - K\phi^2 < B^2$$

when  $\phi \neq 0$ . We obtain  $\phi' = 0$  when  $\phi(t) = C_*$ , where

$$C_*^2 = \frac{-K + \sqrt{B^2 + K^2}}{2}. \quad (3.20)$$

If  $|\phi| > |C_*|$ , then  $B^2 - \phi^4 - K\phi^2 < 0$ , and hence Eq. (3.19) does not have a solution. We have arrived at the following significant results.

**Theorem 3.2.** Provided that the condition (3.17) holds, the solutions of Eq. (3.19) are bounded oscillating functions  $\phi(t)$  satisfying the condition

$$-C_* \leq \phi(t) \leq C_*, \quad (3.21)$$

where  $C_*$  is the positive constant defined by Eq. (3.20). In this notation, the invariant solution (3.15) is written as follows:

$$\begin{aligned} v &= \frac{1}{f} \left[ (2\phi^2(t) + K - f^2) x + 2\phi'(t)z \right], \\ \rho &= \frac{1}{g} \left[ 2\phi'(t)x - (2\phi^2(t) + K - N^2) z \right], \\ \psi &= (x^2 + z^2) \phi(t). \end{aligned} \quad (3.22)$$

**Remark 3.2.** The invariance of the solution (3.15) with respect to rotations (rotational symmetry) means that it has the same values on any circle

$$x^2 + z^2 = r^2$$

with a given radius  $r$ . The invariance under dilations means that we can obtain the solution at any circle just by stretching the radius  $r$ . According to Theorem 3.2, this solution is given by bounded oscillating functions.



## 4 Energy of the rotationally symmetric solution

The conservation of energy for Eqs. (1.1)-(1.3) has the form [1]

$$\frac{d}{dt} \int \int \left[ v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2 \right] dx dz = 0. \quad (4.1)$$

Hence, the *energy density* is

$$E = v^2 + \frac{g^2}{N^2} \rho^2 + |\nabla \psi|^2. \quad (4.2)$$

For the rotationally invariant solution (3.22) we have

$$|\nabla \psi|^2 = 4(x^2 + z^2) \phi^2(t). \quad (4.3)$$

Substituting the expression (4.3) and the expressions (3.22) of  $v$  and  $\rho$  in Eq. (4.2) we obtain the following energy density for the invariant solution (3.22):

$$\begin{aligned} E = & 4 \left( \frac{1}{f^2} - \frac{1}{N^2} \right) (x^2 - z^2) [\phi^2(t) + K] \phi^2(t) + \left( f - \frac{K}{f} \right)^2 x^2 \\ & + \left( N - \frac{K}{N} \right)^2 z^2 + 4 \left( \frac{1}{f^2} - \frac{1}{N^2} \right) xz [2\phi^2(t) + K] \phi'(t). \end{aligned} \quad (4.4)$$

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